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Stable Regions Represented by Time Delays

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1. Introduction

We consider the equations

$$\dot{x}(t) = ax(t-\tau) + bx(t-h) \tag{1}$$

and

$$\dot{x}(t) = ax(t-\tau) + b\int_{t-h}^{t} x(s)ds, \tag{2}$$

where $\tau > 0$ and $h > 0$, and either of a or b is nonzero.

In [3], the authors discussed the region of (a, b) with fixed τ and h for which the zero solution of (2) is asymptotically stable. Our new interest is the region of (τ, h) such that the zero solution of (1) or (2) is asymptotically stable. In what follows, such a region is called stable region.

In case $a = b < 0$, it was shown by Stépán [4] that the zero solution of (1) is asymptotically stable if and only if

$$-a(\tau+h)\cos\left(\frac{\tau-h}{\tau+h}\frac{\pi}{2}\right) < \frac{\pi}{2}.$$

(See Figure 2.1.) Hale and Huang [2] gave the stable region for

$$\dot{x}(t) = ax(t-\tau) + bx(t-h) + cx(t) \tag{1$'$}$$

in case $a \neq b$, assuming that the region is connected. On the other hand, Elsken [1] showed that the unstable region for (1)$'$ in connected.

In section 2 we shall discuss the stable region for (1) in a different way from [2]. In section 3 we shall discuss the stable region for (2) in a similar way to section 2. In both sections, the connectedness of the stable region is assumed.

Let a and b be fixed. For each (τ, h) on the boundary curve of the stable region, the characteristic equation of (1) or (2) has the zero root or a pair of purely imaginary roots $\pm i\omega$. The outline of our method is the following: We shall express τ and h by the multi-valued functions of ω and obtain the boundary curve of the stable region by the parametrized curve $(\tau(\omega), h(\omega))$ in the τh-plane.

2. Stable region for (1)

The characteristic equation for (1) is expressed as

$$\lambda = ae^{-\lambda\tau} + be^{-\lambda h}, \tag{3}$$

where we may assume $|a| \geq |b|$. This equation has the zero root $\lambda = 0$ only if

$$a + b = 0. \tag{4}$$

In order to find the boundary of the stable region, suppose (3) has a pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. Then τ, h and ω must satisfy

$$a \cos \omega\tau + b \cos \omega h = 0$$

and

$$a \sin \omega\tau + b \sin \omega h = -\omega,$$

and so an elementary calculation shows

$$\sin \omega\tau = -\frac{\omega^2 + a^2 - b^2}{2a\omega} \tag{5}$$

and

$$\sin \omega h = -\frac{\omega^2 - a^2 + b^2}{2b\omega}, \tag{6}$$

if $ab \neq 0$. Since $a + b > 0$ implies that (3) has a positive root, we may assume that a and b fulfill the inequality $a + b < 0$. Hence we need only consider three cases:

Case I $\quad a = b < 0$ $\qquad$ Case II $\quad a < b < 0$ $\qquad$ Case III $\quad a < 0 < b < -a$

Now, note that (5) and (6) hold only if ω satisfies the inequalities

$$-1 \leq \frac{\omega^2 + a^2 - b^2}{2a\omega} \leq 1 \tag{7}$$

and

$$-1 \leq \frac{\omega^2 - a^2 + b^2}{2b\omega} \leq 1. \tag{8}$$

On the other hand, (3) implies

$$ab \cos \omega\tau \cdot \cos \omega h = -a^2 \cos^2 \omega\tau = -b^2 \cos^2 \omega h.$$

Hence we have the following proposition.

Proposition 2.1. *If* $\cos \omega\tau \cdot \cos \omega h \neq 0$, *then* $\operatorname{sgn}(ab) = -\operatorname{sgn}(\cos \omega\tau \cdot \cos \omega h)$.

In Case I, both of (7) and (8) mean that ω satisfies

$$0 < \omega \leq -2a.$$

Since $-\frac{\omega^2+a^2-b^2}{2a\omega} = -\frac{\omega}{2a} \to 0$ as $\omega \to +0$, $\omega\tau$ tends to $2n\pi + 0$ or $(2n+1)\pi - 0$ as $\omega \to +0$. Similarly, ωh tends to $2m\pi + 0$ or $(2m+1)\pi - 0$ as $\omega \to +0$. So, by Proposition 2.1, we obtain the curves defined for $\omega \in (0, -2a]$:

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{2n\pi + \operatorname{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\} \\ h &= \frac{1}{\omega}\left\{(2m+1)\pi - \operatorname{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\} \end{aligned} \tag{9}$$

and

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{(2n+1)\pi - \operatorname{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\} \\ h &= \frac{1}{\omega}\left\{2m\pi + \operatorname{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\} \end{aligned} \tag{10}$$

for $n \geq 0$ and $m \geq 0$.

The boundary of the stable region consists of the τ-axis, h-axis and the curves (9), (10) for $n = m = 0$. The stable region for the case of $a = b < 0$ is illustrated by the shaded portion in Figure 2.1.

In Case II, (5) and (6) hold only if

$$-a+b \le \omega \le -a-b.$$

Considering the variation of $\sin\omega\tau$, we have the curves defined for $\omega \in [-a+b, -a-b]$:

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{2n\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega^2+a^2-b^2}{2a\omega}\right)\right\} \\ h &= \frac{1}{\omega}\left\{(2m+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega^2-a^2+b^2}{2b\omega}\right)\right\} \end{aligned} \tag{11}$$

and

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega^2+a^2-b^2}{2a\omega}\right)\right\} \\ h &= \frac{1}{\omega}\left\{2m\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega^2-a^2+b^2}{2b\omega}\right)\right\} \end{aligned} \tag{12}$$

for $n \ge 0$ and $m \ge 0$.

Figures 2.2–2.4 illustrate the stable regions for the case of $a < b < 0$.

In Case III, we have the curves defined for $\omega \in [-a-b, -a+b]$:

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{2n\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega^2+a^2-b^2}{2a\omega}\right)\right\} \\ h &= \frac{1}{\omega}\left\{2m\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega^2-a^2+b^2}{2b\omega}\right)\right\} \end{aligned} \tag{13}$$

and

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega^2+a^2-b^2}{2a\omega}\right)\right\} \\ h &= \frac{1}{\omega}\left\{(2m+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega^2-a^2+b^2}{2b\omega}\right)\right\} \end{aligned} \tag{14}$$

for $n \ge 0$ and $m \ge 0$.

Figures 2.6–2.8 illustrate the stable regions for the case of $a < 0 < b < -a$.

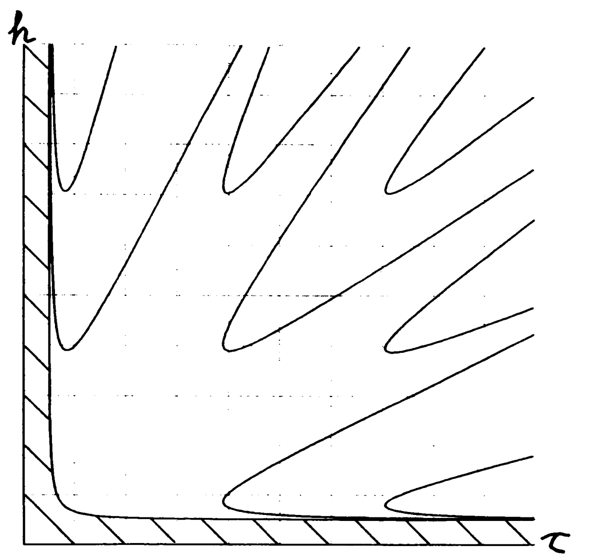

Fig. 2.1 ($a = b = -1$)

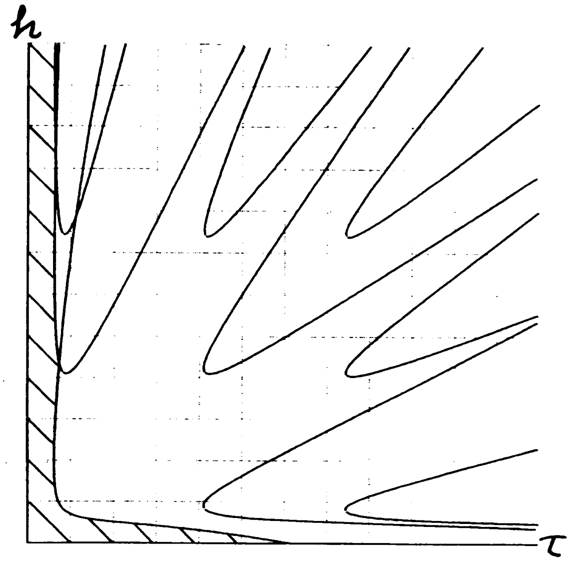

Fig. 2.2 ($a = -1$, $b = -0.9$)

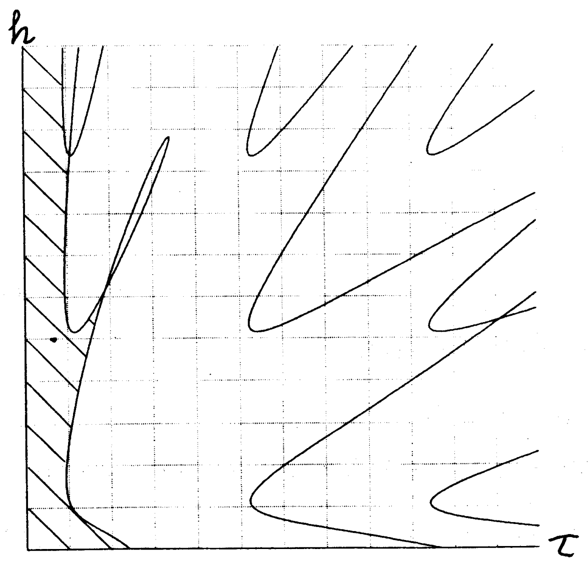

Fig. 2.3 ($a=-1$, $b=-0.5$)

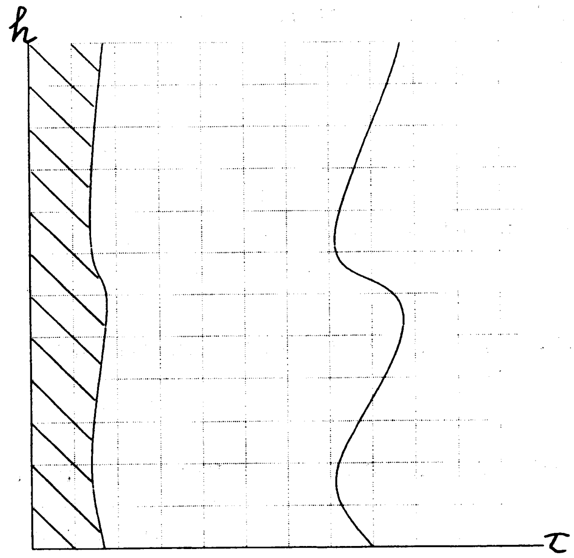

Fig. 2.4 ($a=-1$, $b=-0.1$)

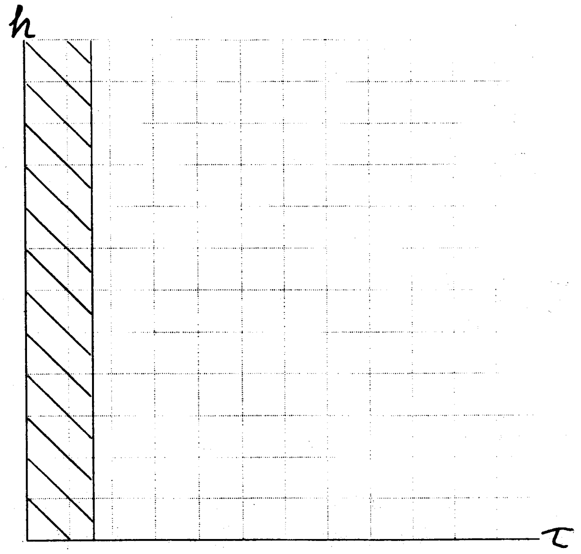

Fig. 2.5 ($a=-1$, $b=0$)

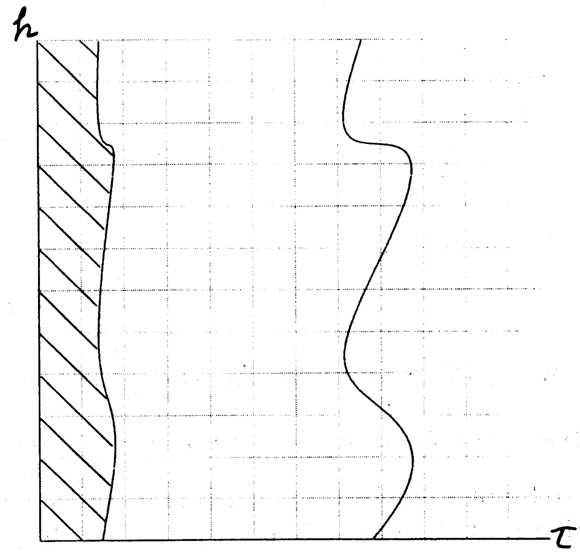

Fig. 2.6 ($a=-1$, $b=0.1$)

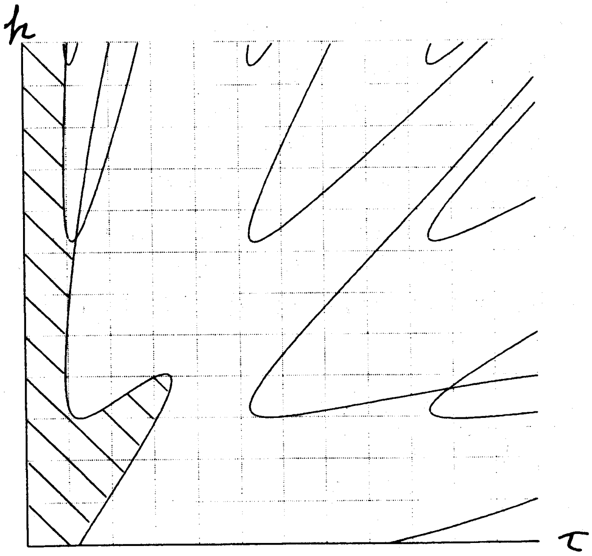

Fig. 2.7 ($a=-1$, $b=0.5$)

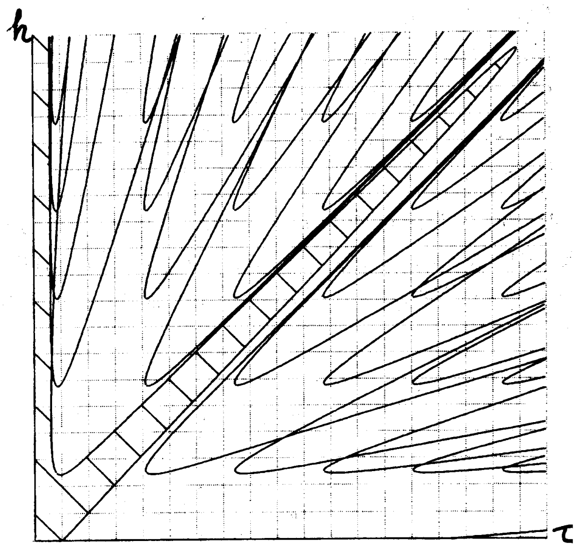

Fig. 2.8 ($a=-1$, $b=0.9$)

3. Stable region for (2)

The characteristic equation for (2) is

$$\lambda = ae^{-\lambda\tau} + b\int_{-h}^{0} e^{\lambda s} ds. \tag{15}$$

This equation has the zero root only if

$$a + bh = 0. \tag{16}$$

On the other hand, if λ is a nonzero root of (15), then λ satisfies

$$\lambda^2 = a\lambda e^{-\lambda\tau} + b(1 - e^{-\lambda h}). \tag{17}$$

Suppose (17) has a pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. Then τ, h and ω satisfy

$$a\omega \sin\omega\tau + b(1 - \cos\omega h) = -\omega^2 \tag{18}$$

and

$$a\omega \cos\omega\tau + b\sin\omega h = 0, \tag{19}$$

and hence

$$\sin\omega\tau = -\frac{\omega(\omega^2 + a^2 + 2b)}{2a(\omega^2 + b)} \tag{20}$$

and

$$\cos\omega h = 1 + \frac{\omega^2(\omega^2 - a^2)}{2b(\omega^2 + b)}. \tag{21}$$

Moreover, (19) ensures the following proposition.

Proposition 3.1. *If* $\cos\omega\tau \cdot \sin\omega h \neq 0$, *then* $\mathrm{sgn}(ab) = -\mathrm{sgn}(\cos\omega\tau \cdot \sin\omega h)$.

If both of a and b are nonnegative, then $a + bh > 0$ for any $h > 0$, and so (15) has a positive root. Therefore we may assume that either of a or b is negative. So, we classify sets of a and b into ten cases:

Case 1	$a < 0$,	$8b \geq a^2$	Case 6	$a < 0$,	$b < -a^2$
Case 2	$a < 0$,	$0 < 8b < a^2$	Case 7	$a = 0$,	$b < 0$
Case 3	$a < 0$,	$b = 0$	Case 8	$a > 0$,	$b < -a^2$
Case 4	$a < 0$,	$-a^2 < b < 0$	Case 9	$a > 0$,	$b = -a^2$
Case 5	$a < 0$,	$b = -a^2$	Case 10	$a > 0$,	$-a^2 < b < 0$

Now we shall find the curves in the τh-plane such that for any (τ, h) lying on any one of those curves, (15) has a pair of purely imaginary roots $\pm i\omega$, $\omega > 0$. In case $\omega^2 + b \neq 0$, according to Proposition 3.1, a calculation shows that τ and h are expressed as follows:

Case 1

$$\begin{aligned} \tau &= \frac{1}{\omega}\left\{2n\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega(\omega^2 + a^2 + 2b)}{2a(\omega^2 + b)}\right)\right\} \\ h &= \frac{1}{\omega}\left\{2m\pi + \mathrm{Cos}^{-1}\left(1 + \frac{\omega^2(\omega^2 - a^2)}{2b(\omega^2 + b)}\right)\right\} \end{aligned} \tag{22}$$

$$\text{for } 0 < \omega \leq -a,\ n \geq 0,\ m \geq 0$$

$$\tau = \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega(\omega^2+a^2+2b)}{2a(\omega^2+b)}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{2m\pi - \mathrm{Cos}^{-1}\left(1+\frac{\omega^2(\omega^2-a^2)}{2b(\omega^2+b)}\right)\right\} \tag{23}$$
$$\text{for } 0<\omega\le -a,\ n\ge 0,\ m\ge 1$$

Case 2

$$\text{curve (22)} \quad \text{for } 0<\omega\le \frac{-a-\sqrt{a^2-8b}}{2},\ n\ge 0,\ m\ge 0$$

or

$$\text{curve (23)} \quad \text{for } 0<\omega\le \frac{-a-\sqrt{a^2-8b}}{2},\ n\ge 0,\ m\ge 1$$

or

$$\text{curve (22)} \quad \text{for } \frac{-a-\sqrt{a^2-8b}}{2}\le\omega\le -a,\ n\ge 0,\ m\ge 0$$

or

$$\text{curve (23)} \quad \text{for } \frac{-a-\sqrt{a^2-8b}}{2}\le\omega\le -a,\ n\ge 0,\ m\ge 0$$

Case 3

$$\tau = -\frac{\pi}{2a}, \quad h>0$$

Case 4

$$\tau = \frac{1}{\omega}\left\{2n\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega(\omega^2+a^2+2b)}{2a(\omega^2+b)}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{2m\pi - \mathrm{Cos}^{-1}\left(1+\frac{\omega^2(\omega^2-a^2)}{2b(\omega^2+b)}\right)\right\} \tag{24}$$
$$\text{for } 0<\omega\le \frac{a+\sqrt{a^2-8b}}{2},\ n\ge 1,\ m\ge 1$$

or

$$\tau = \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega(\omega^2+a^2+2b)}{2a(\omega^2+b)}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{2m\pi + \mathrm{Cos}^{-1}\left(1+\frac{\omega^2(\omega^2-a^2)}{2b(\omega^2+b)}\right)\right\} \tag{25}$$
$$\text{for } 0<\omega\le \frac{a+\sqrt{a^2-8b}}{2},\ n\ge 0,\ m\ge 0$$

or

$$\text{curve (24)} \quad \text{for } -a\le\omega\le \frac{-a+\sqrt{a^2-8b}}{2},\ n\ge 0,\ m\ge 1$$

or

$$\text{curve (25)} \quad \text{for } -a\le\omega\le \frac{-a+\sqrt{a^2-8b}}{2},\ n\ge 0,\ m\ge 0$$

Case 5

$$\tau = \frac{1}{\omega}\left\{2n\pi + \mathrm{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{2m\pi - \mathrm{Cos}^{-1}\left(1-\frac{\omega^2}{2a^2}\right)\right\} \tag{26}$$
$$\text{for } 0<\omega\le -2a,\ n\ge 0,\ m\ge 1$$

$$\tau = \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega}{2a}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{2m\pi + \mathrm{Cos}^{-1}\left(1-\frac{\omega^2}{2a^2}\right)\right\} \qquad (27)$$
$$\text{for } 0<\omega\leq -2a,\ n\geq 0,\ m\geq 0$$

Case 6

$$\text{curve (24)} \quad \text{for } 0<\omega\leq -a,\ n\geq 0,\ m\geq 1$$

or

$$\text{curve (25)} \quad \text{for } 0<\omega\leq -a,\ n\geq 0,\ m\geq 0$$

or

$$\text{curve (24)} \quad \text{for } \frac{a+\sqrt{a^2-8b}}{2}\leq\omega\leq\frac{-a+\sqrt{a^2-8b}}{2},\ n\geq 1,\ m\geq 0$$

or

$$\text{curve (24)} \quad \text{for } \sqrt{-(a^2+2b)}<\omega\leq\frac{-a+\sqrt{a^2-8b}}{2},\ n=0,\ m\geq 0$$

or

$$\text{curve (25)} \quad \text{for } \frac{a+\sqrt{a^2-8b}}{2}\leq\omega\leq\frac{-a+\sqrt{a^2-8b}}{2},\ n\geq 0,\ m\geq 0$$

Case 7

$$\tau>0, \quad h=\frac{\pi}{\sqrt{-2b}}$$

Case 8

$$\text{curve (22)} \quad \text{for } 0<\omega\leq a,\ n\geq 1,\ m\geq 0$$

or

$$\text{curve (23)} \quad \text{for } 0<\omega\leq a,\ n\geq 0,\ m\geq 1$$

or

$$\text{curve (22)} \quad \text{for } \frac{-a+\sqrt{a^2-8b}}{2}\leq\omega\leq\frac{a+\sqrt{a^2-8b}}{2},\ n\geq 1,\ m\geq 0$$

or

$$\text{curve (22)} \quad \text{for } \sqrt{-(a^2+2b)}<\omega\leq\frac{a+\sqrt{a^2-8b}}{2},\ n=0,\ m\geq 0$$

or

$$\tau = \frac{1}{\omega}\left\{(2n+1)\pi - \mathrm{Sin}^{-1}\left(-\frac{\omega(\omega^2+a^2+2b)}{2a(\omega^2+b)}\right)\right\}$$
$$h = \frac{1}{\omega}\left\{(2m+1)\pi - \mathrm{Cos}^{-1}\left(1+\frac{\omega^2(\omega^2-a^2)}{2b(\omega^2+b)}\right)\right\} \qquad (28)$$
$$\text{for } \frac{-a+\sqrt{a^2-8b}}{2}\leq\omega\leq\frac{a+\sqrt{a^2-8b}}{2},\ n\geq 0,\ m\geq 0$$

Case 9

$$\text{curve (22)} \quad \text{for } 0<\omega\leq 2a,\ n\geq 1,\ m\geq 0$$

or

$$\text{curve (23)} \quad \text{for } 0<\omega\leq 2a,\ n\geq 0,\ m\geq 1$$

Case 10

$$\text{curve (22)} \quad \text{for } 0 < \omega \le \frac{-a + \sqrt{a^2 - 8b}}{2}, \quad n \ge 0, \quad m \ge 0$$

or

$$\text{curve (23)} \quad \text{for } 0 < \omega \le \frac{-a + \sqrt{a^2 - 8b}}{2}, \quad n \ge 0, \quad m \ge 1$$

or

$$\text{curve (22)} \quad \text{for } a \le \omega \le \frac{a + \sqrt{a^2 - 8b}}{2}, \quad n \ge 1, \quad m \ge 0$$

or

$$\text{curve (23)} \quad \text{for } a \le \omega \le \frac{a + \sqrt{a^2 - 8b}}{2}, \quad n \ge 0, \quad m \ge 1$$

On the other hand, in case $\omega^2 + b = 0$, (18) implies

$$a\omega \sin \omega\tau - b \cos \omega h = 0. \tag{29}$$

Then it follows from (19) that

$$b \cos(\omega h - \omega\tau) = 0, \tag{30}$$

$$a\omega \sin(\omega h - \omega\tau) = -b \tag{31}$$

and

$$b \sin(\omega h - \omega\tau) = -a\omega. \tag{32}$$

Since $\omega^2 = -b$, (31) and (32) yield

$$\omega^2 = a^2 = -b.$$

Thus, $\omega^2 + b = 0$ holds only if $b = -a^2$. Moreover, we have from (30) that if $\omega^2 + b = 0$ then

$$\omega h = \omega\tau + \frac{\pi}{2} \cdot \operatorname{sgn} a + 2k\pi$$

for some integer k. Finally, we need to note that the characteristic equation (15) has the zero root for $h = -a/b$ when $ab < 0$.

Figures 3.1–3.11 illustrate the stable regions for (2). In addition, the stable region for the case of $a = 7$, $b = -4$ is empty.

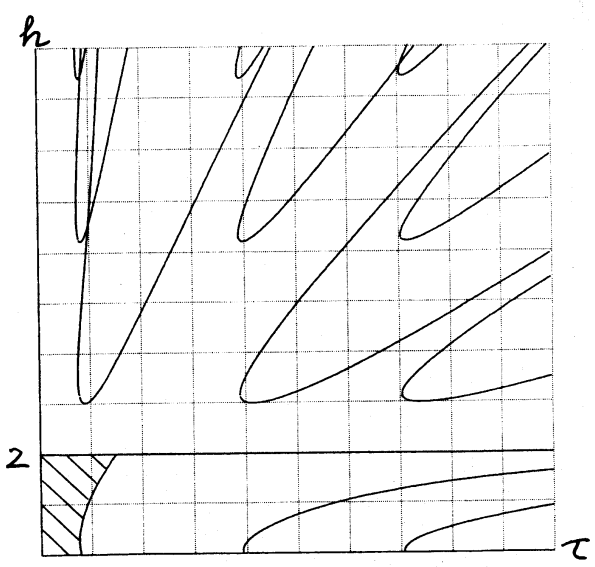

Fig. 3.1 (Case 1; $a = -2$, $b = 1$)

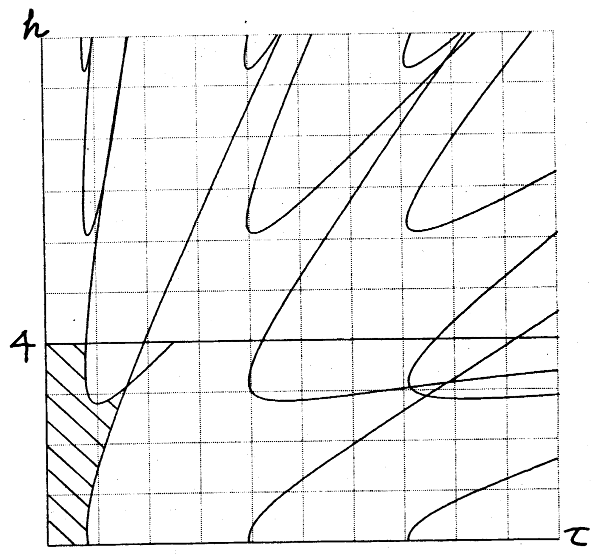

Fig. 3.2 (Case 1; $a = -2$, $b = \frac{1}{2}$)

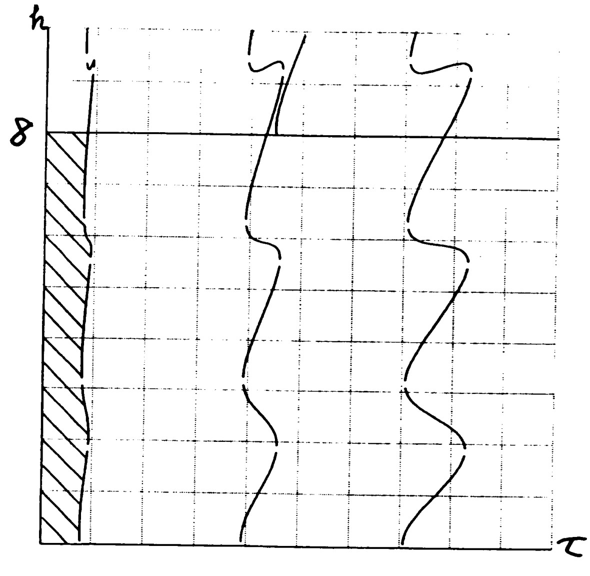

Fig. 3.3 (Case 2; $a = -2$, $b = \frac{1}{4}$)

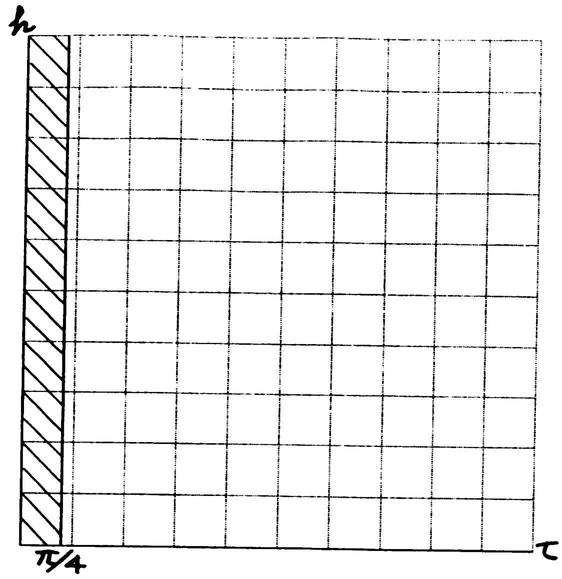

Fig. 3.4 (Case 3; $a = -2$, $b = 0$)

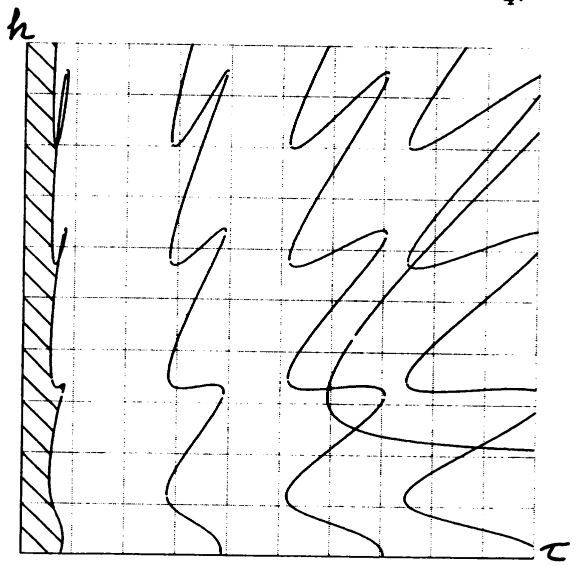

Fig. 3.5 (Case 4; $a = -2$, $b = -1$)

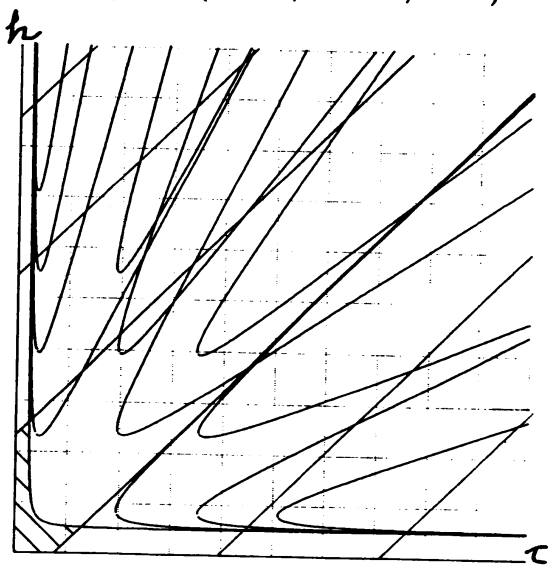

Fig. 3.6 (Case 5; $a = -2$, $b = -4$)

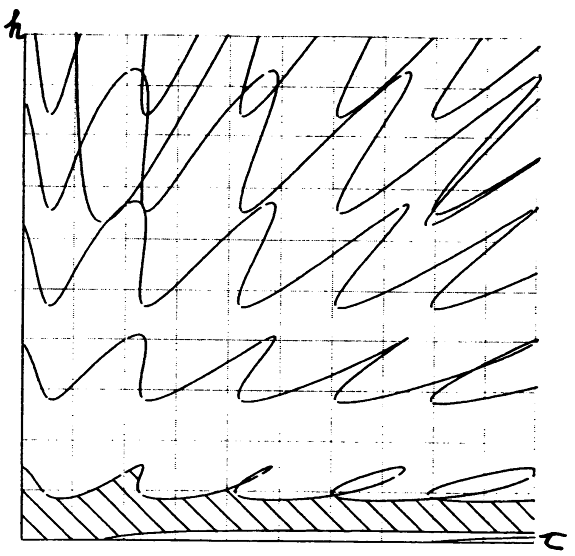

Fig. 3.7 (Case 6; $a = -1$, $b = -4$)

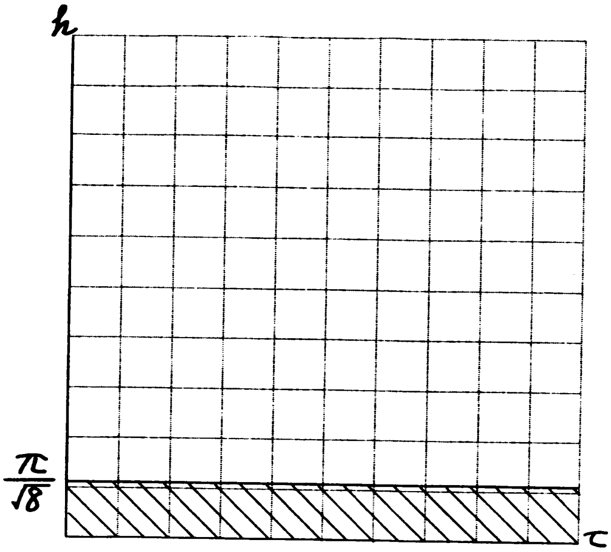

Fig. 3.8 (Case 7; $a = 0$, $b = -4$)

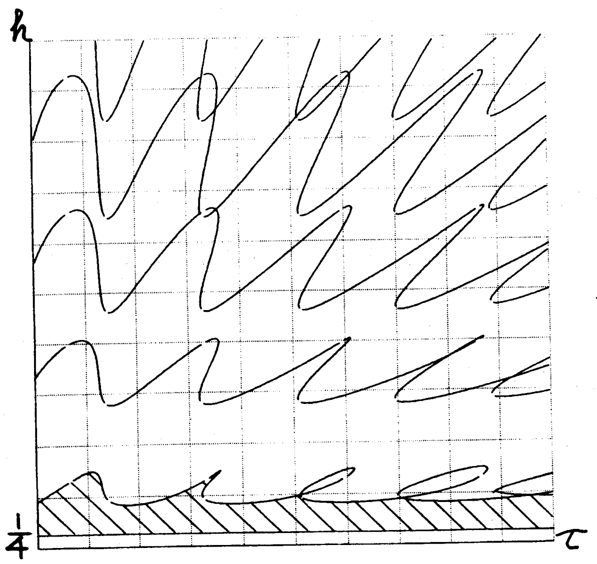

Fig. 3.9 (Case 8; $a = 1,\ b = -4$)

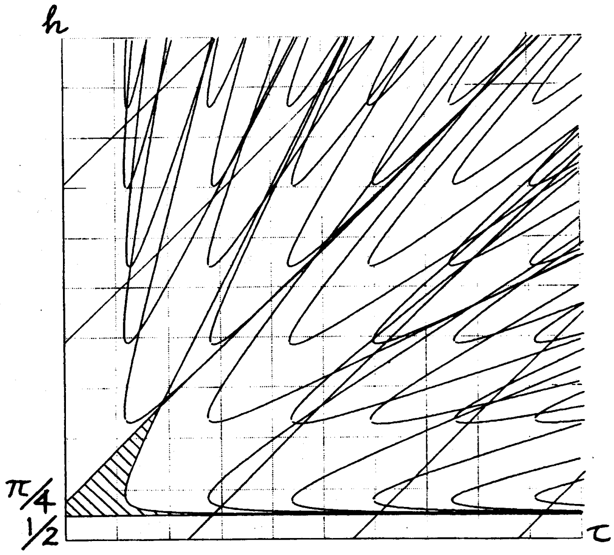

Fig. 3.10 (Case 9; $a = 2,\ b = -4$)

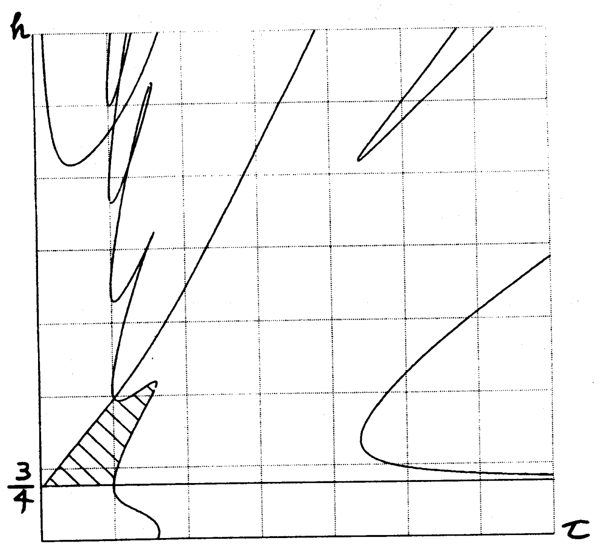

Fig. 3.11 (Case 10; $a = 3,\ b = -4$)

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